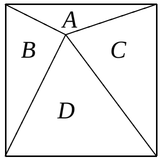
HUT, TUT, UO / Departments of Architecture

Entrance examination in mathematics May 17, 2004

Instructions. Reserve a *separate page* for each problem. Give your solutions in a clear form *including intermediate steps*. Write a clean copy of the solution if needed. *Cross out discarded solutions* (in case of several solutions of the same problem, only the weakest one will be credited).

1. The corners of a square are at the points $(0,0)$, $(1,0)$, $(1,1)$ and $(0,1)$. A point P is chosen in the interior of the square and connected to each of the corners with a line segment. In this way four triangles are obtained with areas A, B, C and D, as in the figure. Determine the coordinates of P if the ratios of the areas satisfy



$$A : B : C : D = 1 : 2 : 3 : 4.$$

2. The degree of urbanization (the percentage of the population living in cities) in Europe was 64.6 % in 1970 but already 74.0 % in 2004. Assume that the change Δp in the degree p of urbanization during the time interval Δt (given in years) satisfies the approximation

$$\Delta p = c \cdot p_0 \cdot (100 - p_0) \cdot \Delta t,$$

where p_0 is the degree of urbanisation at the beginning of the time interval and $c > 0$ is a constant (i.e. it is the same for all time intervals).

   a) Determine the constant c using the time interval 1970–2004.

   b) Give the estimate of the degree of urbanization in Europe at the end of the time interval 2004–2030 based on this model.

3. The straight line $y = 2 - 2x$ and the coordinate axes bound a right-angled triangle. Which point (x_0, y_0) on the hypotenuse of the triangle is at the same distance from the horizontal side as from the midpoint of the vertical side?

4. The Sun shines on a solid standing on a horizontal table. The solid has been obtained by removing a spherical segment of height $r/2$ from a ball of radius r, and the cutting surface is against the table.

   a) How large can the angle of altitude of the Sun be at most if the light rays are to hit at least some point at the base circle of the solid?

   b) How far from the midpoint of the base does the shadow of the solid extend, if the angle of altitude of the Sun is $45°$?

   We may assume that the light rays are parallel. The angle of altitude is the acute angle between the horizontal plane and the light rays.

5. The duration of exams at a carefree university is decided in the following fashion: first the starting time t_0 is randomly drawn from the whole hours $\{9, 10, 11\}$ and then the time of termination is randomly drawn from the whole hours $\{t_0 + 1, \ldots, 13\}$.

   a) What is the probability that an exam will last from 9 o'clock a.m. to 12 o'clock noon?

   b) What is the probability that an exam will last for at least three hours?

6. Three identical balls with radii 3 cm are standing on a table. Each of the balls touches the other two. We try to hide the balls under a hemispherical cover. The cover, however, turns out to be too small, since its edge remains all the way around 1 cm above the table. What is the radius of the cover?

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Solutions

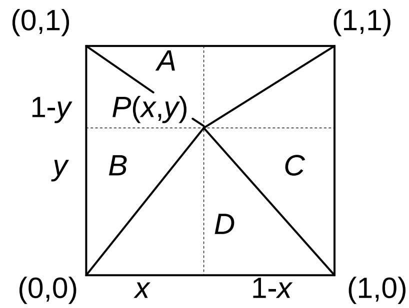
1. Denote $P=(x,y)$. Then, for the areas of the triangles we can read:

for $A = \frac{1}{2}\cdot 1\cdot(1-y)=\frac{1}{2}(1-y)$

for $B = \frac{1}{2}\cdot 1\cdot x=\frac{1}{2}x$

for $C = \frac{1}{2}\cdot 1\cdot(1-x)=\frac{1}{2}(1-x)$

for $D = \frac{1}{2}\cdot 1\cdot y=\frac{1}{2}y$.



Since $A:B:C:D=1:2:3:4$, we have $B:C=\frac{1}{2}x:\frac{1}{2}(1-x)=2:3$, and thus $x=\frac{2}{5}$. In the same way $A:B=\frac{1}{2}(1-y):\frac{1}{2}x=1:2$, or $y=1-\frac{1}{2}x$. Hence $y=1-\frac{1}{2}\cdot\frac{2}{5}=\frac{4}{5}$. Then, the point *P* is $(\frac{2}{5},\frac{4}{5})$. By a direct calculation it also follows $C:D=3:4$.

2. a) In the time interval 1970-2004 we have $p_0=64.6$, $\Delta p=9.4$ and $\Delta t=34$. The model then gives $9.4=c\cdot 64.6\cdot(100-64.6)\cdot 34$. Therefore $c\approx 1.20896\cdot 10^{-4}\approx 1.21\cdot 10^{-4}$.

b) In the time interval 2004-2030 we have $p_0=74.0$ and $\Delta t=34$. By the known constant c the model gives $\Delta p\approx 1.20896\cdot 10^{-4}\cdot 74.0\cdot(100-74.0)\cdot 26\approx 6.0$. Thus the hunted estimation for p is 74.0+ 6.0 = 80.0%.

3. The distance between the points (x_0,y_0) and $(x_0,0)$ must be equal to the distance between the points (x_0,y_0) and (0,1) (see Fig.). Therefore

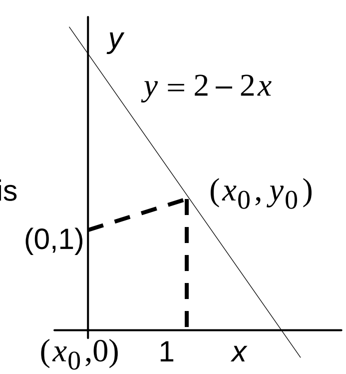
$$y_0=\sqrt{(x_0-0)^2+(y_0-1)^2}\,.$$

This leads to the equation

$$x_0^{\,2}-2y_0+1=0\,.$$

Since the point (x_0,y_0) is lying on the line $y=2-2x$, there is $y_0=2-2x_0$. Hence for x_0 we can write the equation

$$x_0^{\,2}+4x_0-3=0\,.$$

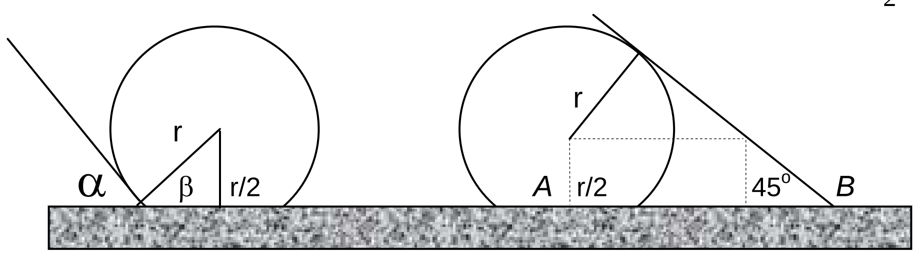


This provides for the solutions $x_0=-2\pm\sqrt{7}$; but only the value of $x_0=-2+\sqrt{7}$ is possible here. Since $y_0=2-2x_0=6-2\sqrt{7}$, for the point (x_0,y_0) we finally get $(-2+\sqrt{7},\,6-2\sqrt{7})$.

4. Consider the solid and light rays as projections on an appropriate vertical plane.

a) The largest angle of altitude α is between the tangent line of the solid and the horizontal plane (see Fig.). Since $\sin\beta=\frac{r/2}{r}=\frac{1}{2}$, then β = 30°. Hence α = 180° - 90° - 30° = 60°.

b) The desired distance is that between the points *A* (the midpoint of the base) and *B* (the most far shadow point) (see Fib.). This evidently is $r\sqrt{2}+r/2=r(\sqrt{2}+\frac{1}{2})$

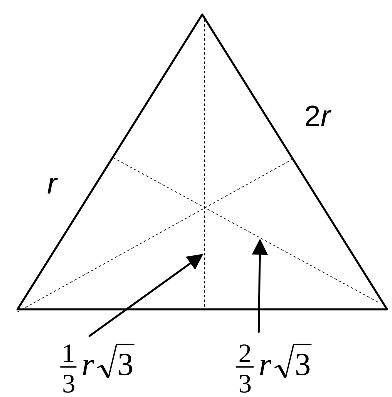


5. a) The probability that an exam starts at 9 o'clock is 1/3. Since the time of termination is drawn randomly from the set $\{10, 11, 12\}$, the probability for the termination time 12 o'clock is also 1/3. Hence the sought probability is (1/3)·(1/3) = 1/9.

b) The proper alternatives for that an exam lasts at least 3 hours are the exam hours of 9-12, 9-13 and 10-13. All of them are separate and independent events. The probability for exam starting times is 1/3. The probability for termination time in the case of the exam hours 9-12 and 9-13 is 1/4 (since random drawing takes place in the set $\{10, 11, 12, 13\}$), and in the case of the exam hour 10-13 is 1/3 (since random drawing takes place in the set $\{11, 12, 13\}$). Hence the required probability is

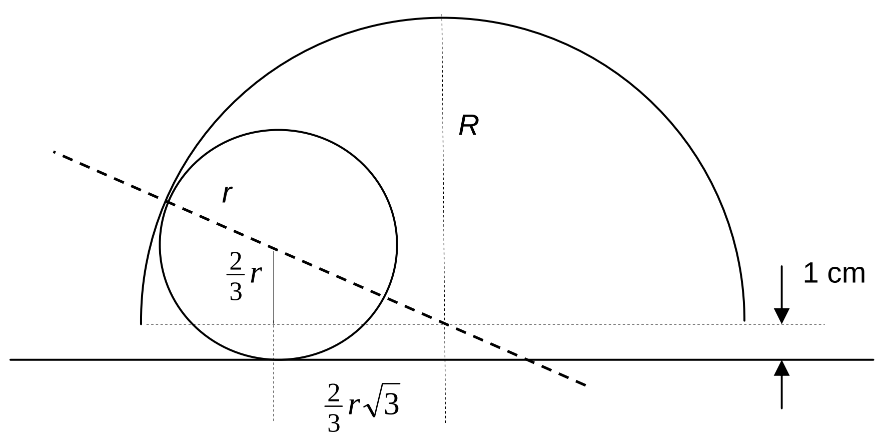
$$\frac{1}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{3} = \frac{5}{18} \approx 0{,}28.$$

6. Let r (r = 3 cm) stand for the radius of the balls. Seen from above, the ceters of the balls locate at the vertices of an equilateral triangle with side lengths 2r and heights $r\sqrt{3}$. Since the common intersection of height segments divides the height segments into parts in the relation of 1:2, the intersection point is found to be in the distance of $(1/3)\cdot r\sqrt{3} = r/\sqrt{3}$ (resp. $(2/3)\cdot r\sqrt{3} = 2r/\sqrt{3}$) from the sides (resp. the vertices) of the triangle. Furthermore, seen from above the intersection point coincides with the midpoint of the cover.



Form next a vertical intersection by a plane which goes through the midpoint of a ball and through the midpoint of the cover:

First, denote by R the radius of the cover, which has to be found (see Fig.).



Since the ball touches the cover inside, the line that passes through the touching point and is the normal line to the tangent plane of the cover at the touching point, passes also through the midpoint of the ball and the midpoint of the cover. Now the intersection figure can be added by a right-angled triangle whose hypotenuse is a part of the radius R. Since the legs of that right-angled triangle are $\frac{2}{3}r\sqrt{3}$ and $\frac{2}{3}r$ in length (here the radius r = 3 cm, so that r -1 cm = $\frac{2}{3}r$), we can finally write for R the result

$$R = r + \sqrt{\left(\frac{2}{3}r\right)^2 + \left(\frac{2}{3}r\sqrt{3}\right)^2} = r + \frac{4}{3}r = \frac{7}{3}r = \frac{7}{3}\cdot 3 \text{ cm} = 7 \text{ cm}.$$